Square Identities

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Here are formulas of roots square identities:

These identities say that the product of 2 numbers, each of which is a sum of 2^n squares, is itself a sum of 2^n squares. But, this is not limited to only 'numbers', and this has many applications :)

Brahmagupta–Fibonacci identity / Diophantus identity

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = z_1^2 + z_2^2 \tag{1}$$

Where,

$$\begin{aligned} z_1 &= (x_1y_1 - x_2y_2) \\ z_2 &= (x_1y_2 + x_2y_1) \end{aligned} \quad \text{or} \quad \begin{aligned} z_1 &= (x_1y_1 + x_2y_2) \\ z_2 &= (x_1y_2 - x_2y_1) \end{aligned} \tag{2}$$

Euler's four–square identity

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2 \tag{3}$$

Where,

$$\begin{aligned}
z_1 &= (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4) \\
z_2 &= (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3) \\
z_3 &= (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2) \\
z_4 &= (x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)
\end{aligned} \tag{4}$$

Degen's eight–square identity

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 + z_7^2 + z_8^2 \tag{5}$$

Where,

$$\begin{aligned}
z_1 &= (x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8) \\
z_2 &= (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5 - x_7y_8 + x_8y_7) \\
z_3 &= (x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 + x_5y_7 + x_6y_8 - x_7y_5 - x_8y_6) \\
z_4 &= (x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1 + x_5y_8 - x_6y_7 + x_7y_6 - x_8y_5) \\
z_5 &= (x_1y_5 - x_2y_6 - x_3y_7 - x_4y_8 + x_5y_1 + x_6y_2 + x_7y_3 + x_8y_4) \\
z_6 &= (x_1y_6 + x_2y_5 - x_3y_8 + x_4y_7 - x_5y_2 + x_6y_1 - x_7y_4 + x_8y_3) \\
z_7 &= (x_1y_7 + x_2y_8 + x_3y_5 - x_4y_6 - x_5y_3 + x_6y_4 + x_7y_1 - x_8y_2) \\
z_8 &= (x_1y_8 - x_2y_7 + x_3y_6 + x_4y_5 - x_5y_4 - x_6y_3 + x_7y_2 + x_8y_1)
\end{aligned} \tag{6}$$

Pfister's sixteen–square identity

$$(x_1^2 + x_2^2 + x_3^2 + \cdots + x_{16}^2)\,(y_1^2 + y_2^2 + y_3^2 + \cdots + y_{16}^2) = z_1^2 + z_2^2 + z_3^2 + \cdots + z_{16}^2 \tag{7}$$

Where,

$$\begin{aligned}
z_1 &= x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8 + u_1y_9 - u_2y_{10} - u_3y_{11} - u_4y_{12} - u_5y_{13} - u_6y_{14} - u_7y_{15} - u_8y_{16} \\
z_2 &= x_2y_1 + x_1y_2 + x_4y_3 - x_3y_4 + x_6y_5 - x_5y_6 - x_8y_7 + x_7y_8 + u_2y_9 + u_1y_{10} + u_4y_{11} - u_3y_{12} + u_6y_{13} - u_5y_{14} - u_8y_{15} + u_7y_{16} \\
z_3 &= x_3y_1 - x_4y_2 + x_1y_3 + x_2y_4 + x_7y_5 + x_8y_6 - x_5y_7 - x_6y_8 + u_3y_9 - u_4y_{10} + u_1y_{11} + u_2y_{12} + u_7y_{13} + u_8y_{14} - u_5y_{15} - u_6y_{16} \\
z_4 &= x_4y_1 + x_3y_2 - x_2y_3 + x_1y_4 + x_8y_5 - x_7y_6 + x_6y_7 - x_5y_8 + u_4y_9 + u_3y_{10} - u_2y_{11} + u_1y_{12} + u_8y_{13} - u_7y_{14} + u_6y_{15} - u_5y_{16} \\
z_5 &= x_5y_1 - x_6y_2 - x_7y_3 - x_8y_4 + x_1y_5 + x_2y_6 + x_3y_7 + x_4y_8 + u_5y_9 - u_6y_{10} - u_7y_{11} - u_8y_{12} + u_1y_{13} + u_2y_{14} + u_3y_{15} + u_4y_{16} \\
z_6 &= x_6y_1 + x_5y_2 - x_8y_3 + x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8 + u_6y_9 + u_5y_{10} - u_8y_{11} + u_7y_{12} - u_2y_{13} + u_1y_{14} - u_4y_{15} + u_3y_{16} \\
z_7 &= x_7y_1 + x_8y_2 + x_5y_3 - x_6y_4 - x_3y_5 + x_4y_6 + x_1y_7 - x_2y_8 + u_7y_9 + u_8y_{10} + u_5y_{11} - u_6y_{12} - u_3y_{13} + u_4y_{14} + u_1y_{15} - u_2y_{16} \\
z_8 &= x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 - x_4y_5 - x_3y_6 + x_2y_7 + x_1y_8 + u_8y_9 - u_7y_{10} + u_6y_{11} + u_5y_{12} - u_4y_{13} - u_3y_{14} + u_2y_{15} + u_1y_{16} \\
z_9 &= x_9y_1 - x_{10}y_2 - x_{11}y_3 - x_{12}y_4 - x_{13}y_5 - x_{14}y_6 - x_{15}y_7 - x_{16}y_8 + x_1y_9 - x_2y_{10} - x_3y_{11} - x_4y_{12} - x_5y_{13} - x_6y_{14} - x_7y_{15} - x_8y_{16} \\
z_{10} &= x_{10}y_1 + x_9y_2 + x_{12}y_3 - x_{11}y_4 + x_{14}y_5 - x_{13}y_6 - x_{16}y_7 + x_{15}y_8 + x_2y_9 + x_1y_{10} + x_4y_{11} - x_3y_{12} + x_6y_{13} - x_5y_{14} - x_8y_{15} + x_7y_{16} \\
z_{11} &= x_{11}y_1 - x_{12}y_2 + x_9y_3 + x_{10}y_4 + x_{15}y_5 + x_{16}y_6 - x_{13}y_7 - x_{14}y_8 + x_3y_9 - x_4y_{10} + x_1y_{11} + x_2y_{12} + x_7y_{13} + x_8y_{14} - x_5y_{15} - x_6y_{16} \\
z_{12} &= x_{12}y_1 + x_{11}y_2 - x_{10}y_3 + x_9y_4 + x_{16}y_5 - x_{15}y_6 + x_{14}y_7 - x_{13}y_8 + x_4y_9 + x_3y_{10} - x_2y_{11} + x_1y_{12} + x_8y_{13} - x_7y_{14} + x_6y_{15} - x_5y_{16} \\
z_{13} &= x_{13}y_1 - x_{14}y_2 - x_{15}y_3 - x_{16}y_4 + x_9y_5 + x_{10}y_6 + x_{11}y_7 + x_{12}y_8 + x_5y_9 - x_6y_{10} - x_7y_{11} - x_8y_{12} + x_1y_{13} + x_2y_{14} + x_3y_{15} + x_4y_{16} \\
z_{14} &= x_{14}y_1 + x_{13}y_2 - x_{16}y_3 + x_{15}y_4 - x_{10}y_5 + x_9y_6 - x_{12}y_7 + x_{11}y_8 + x_6y_9 + x_5y_{10} - x_8y_{11} + x_7y_{12} - x_2y_{13} + x_1y_{14} - x_4y_{15} + x_3y_{16} \\
z_{15} &= x_{15}y_1 + x_{16}y_2 + x_{13}y_3 - x_{14}y_4 - x_{11}y_5 + x_{12}y_6 + x_9y_7 - x_{10}y_8 + x_7y_9 + x_8y_{10} + x_5y_{11} - x_6y_{12} - x_3y_{13} + x_4y_{14} + x_1y_{15} - x_2y_{16} \\
z_{16} &= x_{16}y_1 - x_{15}y_2 + x_{14}y_3 + x_{13}y_4 - x_{12}y_5 - x_{11}y_6 + x_{10}y_7 + x_9y_8 + x_8y_9 - x_7y_{10} + x_6y_{11} + x_5y_{12} - x_4y_{13} - x_3y_{14} + x_2y_{15} + x_1y_{16}
\end{aligned} \tag{8}$$