Coded Computing for Straggler Mitigation, Security and Privacy

EE605 Error Correcting Codes

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Autumn 2021-22

Outline

Distributed Matrix Multiplication

Problem formulation

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- Each of the N workers stores $\frac{1}{m}$ fraction of A and $\frac{1}{n}$ fraction of B where $m, n \in \mathbb{N}$
- Thus $A_i \in \mathbb{F}_q^{s \times \frac{r}{m}}$ and $B_i \in \mathbb{F}_q^{s \times \frac{t}{n}}$

Idea is for the master to use something like an MDS encoding to generate each of these sub matrices such that it has to wait only for k of these workers(fastest) to generate the output in order to uniquely identify the product.

Problem Formulation

Key Ideas

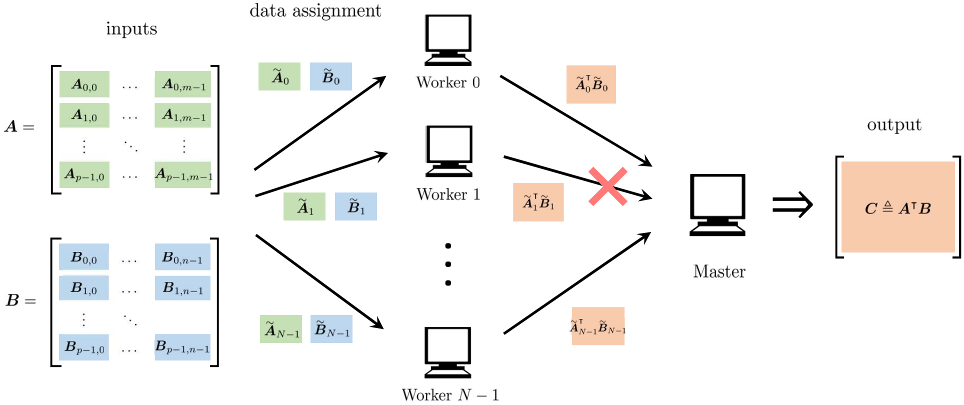

Figure: Overview of the distributed matrix multiplication problem[1].

[1] Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, "Straggler mitigation in distributed matrix multiplication: Fundamental limits and optimal coding," 2020.

Computation Strategy

Choice of functions

A computation strategy is defined as a set of $2N$ functions

$$f = (f_0, f_1.....f_{N-1})$$

$$g = (g_0, g_1.....g_{N-1})$$

that are used to compute $A_i = f_i(A)$, $B_i = g_i(B)\ \forall i \in \{0, 1, 2, ...N-1\}$

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We define $k(f, g)$ as the least integer k for which the system defined by f, g is k recoverable

Computation Strategy

Optimum Recovery Threshold

The lowest among the recovery thresholds across all computation strategies

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The lowest among the recovery thresholds across all computation strategies

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State-of-the-art schemes include

- 1D MDS scheme

$$K_{1D\ MDS} = N - \frac{N}{n} + m$$

- In the same paper an alternative scheme for the special case of $m = n$ referred to as the product code achieves a threshold of

$$K_{prod} = 2(m-1)\sqrt{N} - (m-1)^2 + 1$$

Main Result

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Theorem

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Further $\exists$ a computation strategy referred to as polynomial code which achieves the above K^* which allows for efficient decoding at the master node with computational complexity of polynomial decoding with mn points.

Decoding polynomials codes essentially a polynomial interpolation problem, which can be solved in time almost linear to the input size.This is enabled by designing the computing strategies such that the computed products form a Reed-Solomon code.

Comparison of the four thresholds

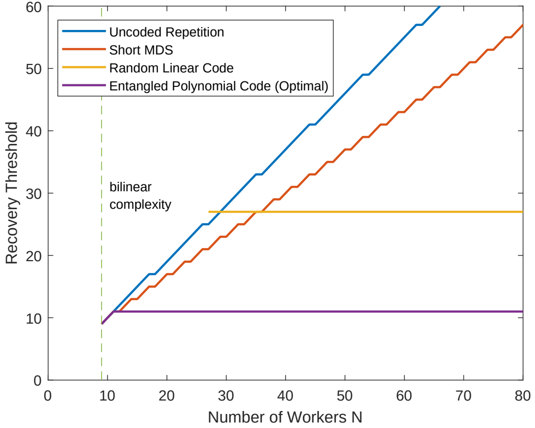

Figure: Comparison[2]

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A sketch of the proof

Given parameters $\alpha, \beta \in \mathbb{N}$, we define the (α, β)- polynomial code $\forall i \in \{0, 1 \ldots N-1\}$ as

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$$\overline{A_i} = \sum_{j=0}^{m-1} A_j x_i^{j\alpha}$$

$$\overline{B_i} = \sum_{j=0}^{n-1} B_j x_i^{j\beta}$$

$$\overline{C_i} = \overline{A_i}^T \overline{B_i} = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} A_j^T B_k x_i^{j\alpha + k\beta}$$

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Note here we need to carefully choose α and β such that no two terms have the same power of x. One such choice being $\alpha = 1$, $\beta = m$. we define $h(x)$ as follows

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This is polynomial of degree $mn - 1$. Since x_i are chosen to be different in order to recover C we need the output from any mn workers which is essentially interpolating using mn points, For even less complex decoding we may use the Reed Solomon decoding algorithm.

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So far we have a scheme that achieves the bound in the theorem. We need to prove that we require output from atleast mn workers to recover C in order to prove optimality

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- Thus one can check the distribution of $C = A^T B$ will be uniform over $\mathbb{F}_q^{r \times t}$
- This means we need to recover a random variable with entropy $H(C) = rt \log q$
- Since each worker outputs $\frac{rt}{mn}$ elements of $\mathbb{F}_q$ it provides atmost $\frac{rt}{mn} \log q$ bits of information

  hence $K > mn$

Performance of the polynomial code on other evaluation metrics

- **Computation latency** is defined as the amount of time required for the master to collect enough information to decode C, For any other computation strategy we have

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- **Communication load** is defined as the minimum number of bits needed to be extracted in order to complete the computation. The below mentioned bound is achieved by the polynomial code.

$$L^* = rt \log_2 q$$

Polynomial Evaluation

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The master waits for a subset of fastest workers and then decodes $Y_1, \ldots, Y_K$.
Linear encoding strategy gives simple yet concrete implmentation,

Some Polynomial Evaluations tasks

Example (Linear Computation)

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Let $\mathbb{V}$ be the space of pairs of two matrices, $\mathbb{U}$ be the space of matrices, $X_i = (A_i, B_i)$, and $f(X_i) = A_i \cdot B_i$ for all $i \in [K]$. (suitable dimensions to be assigned to $\mathbb{V}, \mathbb{U}$)

Result on Minimum Workers required for recovery

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Theorem

Given a number of workers N and a dataset $X = (X_1, \ldots, X_K)$, for distributedly computing f, the minimum recovery threshold is given by

$$K^* = \begin{cases} (K-1)\deg f + 1 & K \deg f - 1 \le N \\ N - \lfloor N/K \rfloor + 1 & \text{else} \end{cases} \tag{1}$$

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Theorem

The Lagrange Coded Computing is optimal i.e. it minimizes the recovery threshold

Proof.

Coming up. □

Polynomial Interpolation

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Given a set of $N + 1$ data points (x_i, y_i) where no two x_i are the same, a polynomial $p : \mathbb{R} \to \mathbb{R}$ is said to interpolate the data if $p(x_j) = y_j$ for each $j \in [N + 1]$

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Construction using System of Linear Equations

Suppose that the interpolation polynomial is in the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

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Now, we can frame a system of linear equations as $p(x_i) = y_i \quad$ for $i \in [N+1]$

$$\begin{bmatrix} x_0^n & x_0^{n-1} & x_0^{n-2} & \dots & x_0 & 1 \\ x_1^n & x_1^{n-1} & x_1^{n-2} & \dots & x_1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_n^n & x_n^{n-1} & x_n^{n-2} & \dots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Lagrange Interpolation[3]

$$p(x) = \frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)} y_0 + \cdots + \frac{(x - x_0)(x - x_1)\cdots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})} y_n$$

[3] https://commons.wikimedia.org/wiki/File:Lagrange_polynomial.svg

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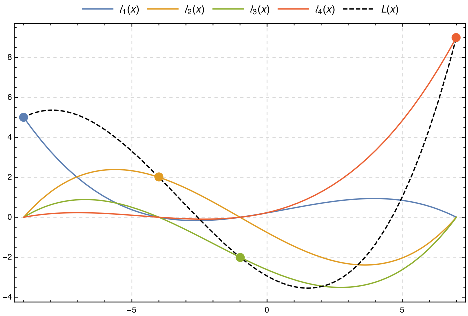
$$p(x) = \sum_{i=0}^{n}\left(\prod_{\substack{0\le j\le n\\ j\ne i}}\frac{x-x_j}{x_i-x_j}\right)y_i \qquad (2)$$



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Key Ideas

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- The computations at the each worker amount to evaluations of a composition of this polynomial with the desired function f resulting in another polynomial h_i.
- Decode $Y_1, \ldots, Y_K$ using only K^* of h_i's by evaluating each at certain points.

Lagrange Coded Computing

Encoding

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Encoding

- Select any K distinct elements $\beta_1, \ldots, \beta_K$ from $\mathbb{F}$, and find a polynomial $u : \mathbb{F} \to \mathbb{V}$ of degree $K-1$ such that $u(\beta_i) = X_i$ for $i \in [K]$.
  This can be accomplished by Lagrange interpolation polynomial

$$u(z) \triangleq \sum_{j\in[K]} X_j \cdot \prod_{k\in[K]\setminus\{j\}} \frac{z-\beta_k}{\beta_j-\beta_k} \quad (3)$$

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- Now, select N distinct elements $\alpha_1, \ldots, \alpha_N$ from $\mathbb{F}$ and encode the input variables by letting $\tilde{X} = u(\alpha_i)$ for $i \in [N]$

$$\tilde{X}_i = g_i(X) = u(\alpha_i) \triangleq \sum_{j\in[K]} X_j \cdot \prod_{k\in[K]\setminus\{j\}} \frac{\alpha_i-\beta_k}{\beta_j-\beta_k} \tag{4}$$

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- Now, any $(K-1)\deg f + 1$ workers return the evaluations at $(K-1)\deg f + 1$ points. This gives a unique $f(u(z))$ which can be interpolated using Lagrange polynomials.

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Note that if, number of workers are small ($N < K\deg f - 1$), K^* can be easily achieved by replicating every X_i by atleast $\lfloor N/K \rfloor$ times. Now, every set of $N - \lfloor N/K \rfloor + 1$ computation contains at least one copy of $f(X_i)$ for every i.

Secure and Private Multiparty Computing

Overview

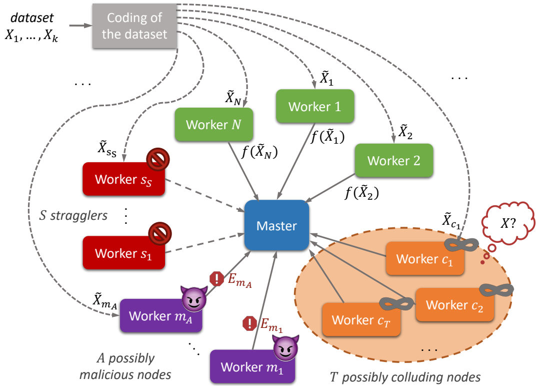

Figure: Overview of Secure and Private Multiparty Computing[4]

[4] J. S. Qian Yu, Netanel Raviv and A. S. Avestimehr, "Lagrange coded computing: Optimal design for resiliency, security and privacy," 2018

Secure and Private Multiparty Computing

Terminology

Secure and Private Multiparty Computing

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- **Resiliency** (robustness against stragglers) In a S-*resilient* system, the master must be able to obtain the correct values of $Y_1, \ldots, Y_K$ even if up to S workers delay/fail.

Secure and Private Multiparty Computing

Terminology

- **Resiliency** (robustness against stragglers) In a S-*resilient* system, the master must be able to obtain the correct values of $Y_1, \ldots, Y_K$ even if up to S workers delay/fail.
- **Security** (robustness against adversaries) In a A-*secure* system, the master must be able to obtain correct values of $Y_1, \ldots, Y_K$ even if up to A workers return arbitrarily erroneous results.

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- **Privacy** (robustness against collusion) In a *T-private* system, the workers cannot infer anything about the content of the dataset, even if up to T of them collude, Formally, for every $\mathcal{T} \subseteq [N]$ of size at most T, we must have $I(X; \tilde{X}_{\mathcal{T}}) = 0$

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The tuple (S, A, T) is achievable if there exists an encoding and decoding scheme that can complete the computations in the presence of up to S stragglers, up to A adversarial workers, whilst keeping the dataset private against sets of up to T colluding workers.

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Given a number of workers N and a dataset $X = (X_1, \ldots, X_K)$, LCC provides an S-resilient, A-secure, and T-private scheme for computing $\{f(X_i)\}_{i=1}^{K}$ for any polynomial f, as long as

$$(K + T - 1)\deg f + S + 2A + 1 \leq N. \tag{5}$$

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Interesting Fact

One additional worker can increase its resiliency to stragglers by 1, or increase its robustness to adversaries by 1/2, while maintaining the privacy constraint. Sounds familiar?

Lagrange Coded Computing

Encoding

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Encoding

- Select any $K + T$ distinct elements $\beta_1, \ldots, \beta_{K+T}$ from $\mathbb{F}$, and find a polynomial $u : \mathbb{F} \to \mathbb{V}$ of degree $K + T - 1$ such that $u(\beta_i) = X_i$ for $i \in [K]$ and $u(\beta_i) = Z_i$ for $i \in \{K+1, \ldots, K+T\}$ where are Z_i's are chosen randomly from $\mathbb{V}$.

$$u(z) \triangleq \sum_{j \in [K]} X_j \cdot \prod_{k \in [K+T] \setminus \{j\}} \frac{z - \beta_k}{\beta_j - \beta_k} + \sum_{j=K+1}^{K+T} Z_j \cdot \prod_{k \in [K+T] \setminus \{j\}} \frac{z - \beta_k}{\beta_j - \beta_k} \qquad (6)$$

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- Now, select N distinct elements $\alpha_1,\ldots,\alpha_N$ from $\mathbb{F}$ such that $\{\alpha_i\}_{i=1}^N\cap\{\beta_j\}_{j=1}^N=\varnothing$ and encode the input variables by letting $\tilde{X}=u(\alpha_i)$ for $i\in[N]$

$$\tilde{X}_i=g_i(X)=u(\alpha_i)=(X_1,\ldots,X_K,Z_{K+1},\ldots,Z_{K+T})\cdot U_i \quad (U\in\mathbb{F}_q^{(K+T)\times N}) \quad (7)$$

Where $U_{ij}\triangleq\prod_{l\in[K+T]\setminus\{i\}}\frac{\alpha_j-\beta_l}{\beta_i-\beta_l}$

Lagrange Coded Computing

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- Then, the master evaluates it at β_i for every $i \in [K]$ to obtain $f(u(\beta_i)) = f(X_i)$,

Hence, we have shown that the above scheme is S-resilient and A-secure.

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- Application to blockchain systems
  - Today's blockchain designs suffer from a trilemma claiming that no blockchain system can simultaneously achieve decentralization, security, and performance scalability.
  - Coded computing can provide an effective approach for overcoming such barriers.

For Further Reading

- Q. Yu, M. A. Maddah-Ali, and A. S. Avestimehr, "Straggler mitigation in distributed matrix multiplication: Fundamental limits and optimal coding," *IEEE Transactions on Information Theory*, vol. 66, no. 3, pp. 1920–1933, 2020.
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